# Sensitivity of the asymptotic behaviour of meta distributions

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**Abstract**

The paper focuses on a class of light-tailed multivariate probability distributions. These are obtained via a transformation of the marginals from a heavy-tailed original distribution. This class was introduced in Balkema et al. (2009). As shown there, for the light-tailed meta distribution the sample clouds, properly scaled, converge onto a deterministic set. The shape of the limit set gives a good description of the relation between extreme observations in different directions. This paper investigates how sensitive the limit shape is to changes in the underlying heavy-tailed distribution. Copulas fit in well with multivariate extremes. By Galambos's Theorem existence of directional derivatives in the upper endpoint of the copula is necessary and sufficient for convergence of the multivariate extremes provided the marginal maxima converge. The copula of the max-stable limit distribution does not depend on the marginals. So marginals seem to play a subsidiary role in multivariate extremes. The theory and examples presented in this paper cast a different light on the significance of marginals. For light-tailed meta distributions the asymptotic behaviour is very sensitive to perturbations of the underlying heavy-tailed original distribution, it may change drastically even when the asymptotic behaviour of the heavy-tailed density is not affected.

**Keywords:** extremes, limit set, limit shape, meta distribution, regular partition, sensitivity.

## 1 Introduction

In recent years meta distributions have been used in several applications of multivariate probability theory, especially in finance. The construction of meta distributions can be illustrated by a simple example. Start with a multivariate spherical $t$ distribution and transform its marginals to be Gaussian. The new distribution has normal marginals. It is called a *meta distribution* with normal marginals based on the *original* $t$ distribution. Since the copula of a multivariate distribution is invariant under strictly increas-

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ing coordinatewise transformations, the original distribution and the meta distribution share their copula and hence have the same dependence structure. The light-tailed meta distribution inherits not only the dependence properties of the original $t$ distribution, but also the asymptotic dependency. These asymptotic properties are of importance to risk theory. They include rank-based measures of tail dependence, and the tail dependence coefficients (see e.g. Chapter 5 in [8]). Vectors with Gaussian densities have asymptotically independent components, whatever the correlation of the Gaussian density. The meta density with standard Gaussian marginals based on a spherically symmetric Student $t$ density has the copula of the $t$ distribution, and the max-stable limit distributions for the coordinatewise maxima also have the same copula. The max-stable limit vectors have dependent components.

However, this recipe for constructing distributions with Gaussian, or more generally, light-tailed marginals, based on a heavy-tailed density with a pronounced dependency structure in the limit, has to be treated with caution. The limit shape of the sample clouds from the meta distribution is affected by perturbations of the original heavy-tailed density, perturbations which are so small that they do not affect the multivariate extreme value behaviour. In going from densities with heavy-tailed marginals to the meta densities with light-tailed marginals the dependence structure of the max-stable limit distribution is preserved by a well-known invariance result in multivariate extreme value theory. In this paper we shall give exact conditions on the severity of changes in the original heavy-tailed distribution which are allowed if one wants to retain the asymptotic behaviour of the coordinatewise extremes. It will be shown that perturbations which are negligible compared to these changes may affect the limit shape of the sample clouds of the light-tailed meta distribution.

Multivariate distribution functions (dfs) have the property that there is a very simple relation between the df of the original vector and the df of the coordinatewise maximum of any number of independent observations from this distribution. One just raises the df to the given power. This makes dfs and in particular copulas ideal tools to handle coordinatewise maxima, and to study their limit behaviour. This rather analytic approach sometimes obscures the probabilistic content of the results. The approach via densities and probability measures on $\mathbb{R}^d$ which is taken in this paper may at first seem clumsy, but it has the advantage that there is a close relation to what one observes in the sample clouds.

Our interest is in extremes. The asymptotic behaviour of sample clouds gives a very intuitive view of multivariate extremes. The limit shape of sample clouds, if it exists, describes the relation between extreme observations in different directions; it indicates in which directions more severe extremes are likely to occur, and how much more extreme these will be. It has been shown in [2] that sample clouds from meta distributions in the *standard set-up*, see below, can be scaled to converge onto a *limit set*. The boundary of this limit set has a simple explicit analytic description. The limit shape of sample clouds from the meta distribution contains no information about the shape of the sample clouds from the original distribution. The results of the present paper support this point.

The aim of the paper is to investigate stability of the shape of the limit set under changes in the original distribution. We look at changes which do not affect the marginals, or at least their asymptotic behaviour. Keeping marginals (asymptotically) unchanged allows us to isolate the role played by the copula. We shall examine how much the original and meta distributions in the standard set-up may be altered without affecting the asymptotic behaviour of the scaled sample clouds. This shows how robust the limit shape is. Then we move on to explore sensitivity. The limit shape of the scaled sample clouds from the light-tailed meta distribution is very sensitive to certain slight perturbations of the original distribution, perturbations which affect the density in particular regions. For heavy-tailed distributions the region around the coordinate planes seems to be most sensitive.

The present paper is a follow-up to [2]. The latter paper contains a detailed analysis of meta densities and gives the motivation and implications of the assumptions in the standard set-up. It presents the derivation and analysis of the limit shape of the sample clouds from the light-tailed meta distribution, and may be consulted for more details on these subjects. In the present paper Section 2 introduces the notation and recalls the relevant definitions and results from [2]. Section 3 is the heart of the paper; here we present details of the constructions which demonstrate robustness and sensitivity of the limit shape and the asymptotics of sample clouds from meta distributions. Concluding remarks are given in Section 4. The appendix contains a few supplementary results, and a summary of notation which the reader may find useful when reading the paper.

# 2 Preliminaries

## 2.1 Definitions and standard set-up

The following definition describes the formal procedure for constructing meta distributions.

**Definition 1.** *Let $G_1, \dots, G_d$ be continuous dfs on $\mathbb{R}$ which are strictly increasing on the intervals $I_i = \{0 < G_i < 1\}$. Consider a random vector $\mathbf{Z}$ in $\mathbb{R}^d$ with df $F$ and continuous marginals $F_i$, $i = 1, \dots, d$. Define the transformation*

$$K(x_1, \dots, x_d) = (K_1(x_1), \dots, K_d(x_d)), \qquad K_i(s) = F_i^{-1}(G_i(s)) \qquad i = 1, \dots, d. \tag{2.1}$$

*The df $G = F \circ K$ is the* meta distribution *(with* marginals *$G_i$) based on the* original *df $F$. The random vector $\mathbf{X}$ is said to be a* meta vector *for $\mathbf{Z}$ (with* marginals *$G_i$) if*

$$\mathbf{Z} \overset{\mathrm{d}}{=} K(\mathbf{X}). \tag{2.2}$$

*The coordinatewise map $K = K_1 \otimes \cdots \otimes K_d$ which maps $\mathbf{x} = (x_1, \dots, x_d) \in I = I_1 \times \cdots \times I_d$ into the vector $\mathbf{z} = (K_1(x_1), \dots, K_d(x_d))$ is called the* meta transformation. ♢

The class of distributions above is too general for our purpose. Hence, we choose to restrict our attention to a subclass by imposing more structure on the original distribution and on the marginals of the meta distribution. The standard set-up of this paper is the same as in [2]. Recall the basic example we started with in the introduction.

The multivariate $t$ density has a simple structure. It is fully characterized by the shape of its level sets, scaled copies of the defining ellipsoid, and by the decay $c/r^{\lambda+d}$ of its tails along rays. The constant $\lambda > 0$ denotes the degrees of freedom, $d$ the dimension of the underlying space, and $c$ is a positive constant depending on the direction of the ray. In the more general setting of the paper, the tails of the density are allowed to decrease as $cL(r)/r^{\lambda+d}$ for some slowly varying function $L$ and the condition of elliptical level sets is replaced by the requirement that the level sets are equal to scaled copies of a fixed bounded convex or star-shaped set (a set $D$ is star-shaped if $\mathbf{z} \in D$ implies $t\mathbf{z} \in D$ for $0 \leq t < 1$). Due to the power decay of the tails, the density $f$ is said to be *heavy-tailed*. Densities with the above properties constitute the class $\mathcal{F}_\lambda$.

**Definition 2.** *The set $\mathcal{F}_\lambda$ for $\lambda > 0$ consists of all positive continuous densities $f$ on $\mathbb{R}^d$ which are asymptotic to a function of the form $f_*(n_D(\mathbf{z}))$ where $f_*(r) = L(r)/r^{(\lambda+d)}$ is a continuous decreasing function on $[0, \infty)$, $L$ varies slowly, and $n_D$ is the gauge function of the set $D$. The set $D$ is bounded, open and star-shaped. It contains the origin and has a continuous boundary.*

The reader may keep in mind the case where $D$ is a convex symmetric set. In that case the gauge function is a norm, and $D$ the unit ball. The normal marginals of the meta density are generalized to include densities whose tails are asymptotic to a *von Mises function*: $g_0(s) \sim e^{-\psi(s)}$ for $s \to \infty$ with *scale function* $a = 1/\psi'$, where $\psi$ is a $C^2$ function with a positive derivative such that

$$\psi(s) \to \infty, \qquad a(s)' \to 0 \quad s \to \infty. \tag{2.3}$$

This condition on the meta marginals ensures that they lie in the maximum domain of attraction of the Gumbel limit law $\exp(-e^{-x})$, $x > 0$; see e.g. Proposition 1.4 in [10]. In this case we say that the meta distribution is *light-tailed*.

**Definition 3.** *In the* standard set-up*, the density $f$ lies in $\mathcal{F}_\lambda$ for some $\lambda > 0$, and $g_0$ is continuous, positive, symmetric, and asymptotic to a von Mises function $e^{-\psi}$. We assume that $\psi$ varies regularly in infinity with exponent $\theta > 0$. The density $g$ is the meta density with marginals $g_0$, based on $f$.*

## 2.2 Convergence of sample clouds

An *$n$-point sample cloud* is the point process consisting of the first $n$ points of a sequence of independent observations from a given distribution, after proper scaling. We write

$$N_n = \{\mathbf{Z}_1/a_n, \ldots, \mathbf{Z}_n/a_n\} \tag{2.4}$$

where $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ are independent observations from the given probability distribution on $\mathbb{R}^d$, and $a_n$ are positive scaling constants. It is custom to write $N_n(A)$ for the number of the points of the sample cloud that fall into the set $A$.

In this section, we discuss the asymptotic behaviour of sample clouds from the original density $f$ and from the associated meta density $g$ in the standard set-up. The difference in the asymptotic behaviour is striking: sample clouds from a heavy-tailed density $f$ converge in distribution to a Poisson point process on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ whereas sample clouds from a light-tailed meta density $g$ tend to have a clearly defined boundary. They converge onto a deterministic set.

### 2.2.1 Convergence for densities in $\mathcal{F}_\lambda$ and measures in $\mathcal{D}_\lambda$

For densities in $\mathcal{F}_\lambda$, $\lambda > 0$, there is a simple limit relation:

$$\frac{f(r_n \mathbf{w}_n)}{f_*(r_n)} \to h(\mathbf{w}), \qquad \mathbf{w}_n \to \mathbf{w} \neq \mathbf{0},\ r_n \to \infty, \tag{2.5}$$

where

$$h(\mathbf{w}) = 1/n_D(\mathbf{w})^{\lambda+d} = \eta(\omega)/r^{\lambda+d}, \qquad r = \|\mathbf{w}\|_2 > 0,\ \omega = \mathbf{w}/r. \tag{2.6}$$

Convergence is uniform and $\mathbf{L}^1$ on the complement of centered balls. If $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ are independent observations from the density $f$ then the sample clouds $N_n$ in (2.4) converge in distribution to the Poisson point process with intensity $h$ weakly on the complement of centered balls for a suitable choice of scaling constants $a_n$.

A probability measure $\pi$ on $\mathbb{R}^d$ varies regularly with scaling function $a(t) \to \infty$ if there is an infinite Radon measure $\rho$ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ such that the finite measures $t\pi$ scaled by $a(t)$ converge to $\rho$ vaguely on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. The measure $\rho$ has the scaling property

$$\rho(rA) = \rho(A)/r^\lambda \qquad r > 0 \tag{2.7}$$

for all Borel sets $A$ in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. The constant $\lambda \geq 0$ is the exponent of regular variation. If it is positive then $\rho$ gives finite mass to the complement of the open unit ball $B$ and weak convergence holds on the complement of centered balls. We shall denote the set of all probability measures which vary regularly with exponent $\lambda > 0$ by $\mathcal{D}_\lambda$. In particular $\pi \in \mathcal{D}_\lambda$ if $\pi$ has a continuous density in $\mathcal{F}_\lambda$. As above for independent observations $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ from a distribution $\pi \in \mathcal{D}_\lambda$ the scaled sample clouds $N_n$ in (2.4) (with scaling constants $a(n)$) converge in distribution to a Poisson point process with mean measure $\rho$ weakly on the complement of centered balls (since the mean measures converge, see e.g. Proposition 3.21 in [10] or Theorem 11.2.V in [5]).

#### 2.2.2 Convergence for meta densities in the standard set-up

Sample clouds from light-tailed meta densities in the standard set-up, under suitable scaling, converge onto a deterministic set, referred to as the *limit set*, in the sense of the following definition.

**Definition 4.** *For a compact set $E$ in $\mathbb{R}^d$, the finite point processes $N_n$* converge onto $E$ *if for open sets $U$ containing $E$, the probability of a point outside $U$ vanishes, $\mathbb{P}\{N_n(U^c) > 0\} \to 0$, and if*

$$\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) > m\} \to 1 \qquad m \geq 1,\ \epsilon > 0,\ \mathbf{p} \in E.$$

We now recall Theorem 2.6 of [2], which characterizes the shape of the limit set for meta distributions in the standard set-up.

**Theorem 2.1.** *Let $f$, $g$ and $g_0 \sim e^{-\psi}$ satisfy the assumptions of the standard set-up. Define*

$$E := E_{\lambda,\theta} = \{\mathbf{u} \in \mathbb{R}^d \mid |u_1|^\theta + \cdots + |u_d|^\theta + \lambda \geq (\lambda + d)\|\mathbf{u}\|_\infty^\theta\}. \tag{2.8}$$

*If $\psi(r_n) \sim \log n$, then for the sequence of independent observations $\mathbf{X}_1, \mathbf{X}_2, \ldots$ from the meta density $g$, the sample clouds $M_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ converge onto $E$.*

### 2.3 Further notation and conventions

In order to ease the exposition, we introduce some additional assumptions and notation.

All univariate dfs are assumed to be continuous and strictly increasing. The dfs $F_0$ and $\tilde{F}_0$ on $\mathbb{R}$ are *tail asymptotic* if

$$\tilde{F}_0(-t)/F_0(-t) \to 1 \qquad (1 - \tilde{F}_0(t))/(1 - F_0(t)) \to 1 \qquad t \to \infty.$$

The sample clouds from a heavy-tailed df $\tilde{F}$ *converge to the point process* $\tilde{N}$ if $\tilde{N}$ is a Poisson point process on $\mathbb{R}^d \setminus \{\mathbf{0}\}$, and if the sample clouds converge to $\tilde{N}$ in distribution weakly on the complement of centered balls, where the scaling constants $c_n$ satisfy $1 - \tilde{F}_0(c_n) \sim 1/n$. Two heavy-tailed dfs $F^*$ and $F^{**}$ *have the same asymptotics* if the marginals are tail asymptotic and if the sample clouds converge to the same point process. The light-tailed dfs $G^*$ and $G^{**}$ *have the same asymptotics* if the marginals are tail asymptotic and the sample clouds converge onto the same compact set $E^*$, with scale factors $b_n$ which satisfy $-\log(1 - G_0(b_n)) \sim \log n$. One may replace a scaling sequence $a_n$ by a sequence asymptotic to $a_n$ without affecting the limit. The scaling of sample clouds is determined up to asymptotic equality by the marginals. Tail asymptotic marginals yield asymptotic scalings.

## 3 Results

We now turn our attention to the main issues of the paper. The aim is to investigate how much the dfs $F$ and $G = F \circ K$ in the standard set-up may be altered without affecting the asymptotic behaviour of the

scaled sample clouds. For simplicity we assume here that the marginal densities of $F$ also all are equal to a positive continuous symmetric density $f_0$. It follows that the components of the meta transformation $K$ are equal:

$$K : \mathbf{x} \mapsto \mathbf{z} = (K_0(x_1), \dots, K_0(x_d)) \qquad K_0 = F_0^{-1} \circ G_0 \qquad K_0(-t) = -K_0(t). \tag{3.1}$$

Let $F^*$ be a df with marginal densities $f_0$. Then $G^* = F^* \circ K$ is the meta distribution based on the df $F^*$ with marginals $g_0$. One can pose the following questions:

(Q.1) If the scaled sample clouds from $G^*$ and from $G$ converge onto the same set $E$, do the scaled sample clouds from $F^*$ converge to the same point process $N$ as those from $F$?

(Q.2) If the scaled sample clouds from $F^*$ and from $F$ converge to the same Poisson point process $N$, do the scaled sample clouds from $G^*$ converge onto the same set $E$ as those from $G$?

The answer to the corresponding questions for coordinatewise maxima and their exponent measures (if we also allow translations) is "Yes". Here, for sample clouds and their limit shape, the answer to both questions is "No". This section contains some counterexamples which will be worked out further in the next two sections.

If we replace $f$ by a weakly asymptotic density $f^* \asymp f$, the asymptotic behaviour of sample clouds from $g^*$ is not affected, since $g^* \asymp g$ (see Lemma 3.2 below), but the scaled sample clouds from $f^*$ obviously need not converge. What if they do?

**Example 1.** Let $\mathbf{Z}$ have a spherical Student $t$ density $f(\mathbf{z}) = f_*(\|\mathbf{z}\|)$ with marginals $f_0$ and limit function $h(\mathbf{w}) = 1/\|\mathbf{w}\|^{\lambda+d}$ with marginals $c/t^{\lambda+1}$. The vector with components $\mathbf{a}_1^T\mathbf{Z}, \dots, \mathbf{a}_d^T\mathbf{Z}$ where $\mathbf{a}_1, \dots, \mathbf{a}_d$ are independent unit vectors will have the same marginals $f_0$ but density $f_*(n_E(\mathbf{z}))$ with elliptic level sets, which are spherical only if the vectors $\mathbf{a}_j$ are orthogonal. There are many star-shaped sets $D$ for which $\tilde{h}(\mathbf{w}) = 1/n_D(\mathbf{w})^{\lambda+d}$ has marginals $c/t^{\lambda+1}$ as above. Probability densities $\tilde{f} \sim f_*(n_D)$ will lie in $\mathcal{F}_\lambda$ and have marginals asymptotic to the Student $t$ marginals $f_0$ above. All these densities are weakly asymptotic to each other, $\tilde{f} \asymp f$. The only difference between the densities is in their copulas. The information of the dependency contained in the set $D$ is preserved in the limit of the sample clouds from the density $\tilde{f}$, but lost in the limit shape $E$ of the sample clouds from the meta density $\tilde{g}$. Surprisingly the information on the shape is lost in the step to the meta density $\tilde{g}$, but the tail exponent $\lambda$ of the marginals is still visible in the limit shape $E$. $\diamondsuit$

What we want to do is fix the marginals $f_0$ and $g_0$ (which determine the meta transformation $K$), and then vary the copula and check the limit behaviour of the sample clouds (where we impose the condition that both converge). We are looking for dfs $F^*$ and $G^*$ with the properties:

(P.1) $F^*$ has marginal densities $f_0$;

(P.2) $G^*$ is the meta distribution based on $F^*$ with marginal densities $g_0$;

(P.3) the scaled sample clouds from $F^*$ converge to a Poisson point process $N^*$;

(P.4) the scaled sample clouds from $G^*$ converge onto a compact set $E^*$.

Moreover one would like $E^*$ to be the set $E_{\lambda,\theta}$ in (2.8), or $N^*$ to have mean measure $\rho^* = \rho$ with intensity $h$ in (2.6). So we either choose $F^*$ to have the same asymptotics as $F$, or $G^*$ to have the same asymptotics as $G$. Note that the four conditions above have certain implications. The mean measure $\rho^*$ of the Poisson point process $N^*$ is an excess measure with exponent $\lambda$, see (2.7), its marginals are equal and symmetric with intensity $\lambda/|t|^{\lambda+1}$ since the marginal densities $f_0$ are equal and symmetric and the scaling constants $c_n$ ensure that $\rho\{w_d \geq 1\} = 1$. The limit set $E^*$ is a subset of the cube $C = [-1,1]^d$ and projects onto the interval $[-1,1]$ in each coordinate, again by our choice of the scaling constants.

The two sections below describe procedures for altering distributions without changing the marginals too much. A block partition is a special kind of partition into coordinate blocks. If the blocks are relatively small then the asymptotics of a distribution do not change if one replaces it by one which gives the same mass to each block. Block partitions are mapped into block partitions by $K$. The mass is preserved, but the size and shape of the blocks may change drastically. The block partitions provide insight in the relation between the asymptotic behaviour of the measures $dF^*$ and $dG^*$. In the second procedure we replace $dF$ by a probability measure $d\tilde{F}$, which agrees outside a bounded set with $d(F+F^o)$ where $F^o$ has lighter marginals than $F$:

$$F_j^o(-t) << F_0(-t) \qquad 1 - F_j^o(t) << 1 - F_0(t) \qquad t \to \infty, \qquad j = 1, \dots, d. \tag{3.2}$$

This condition ensures that $\tilde{F}$ and $F$ have the same asymptotics. The two corresponding light-tailed meta dfs $\tilde{G}$ and $G$ on $\mathbf{x}$-space may have different asymptotics since the scaling constants $b_n^o$ and $b_n$ may be asymptotic even though $G^o$ has lighter tails than $G$. If this is the case, and the scaled sample clouds from $G^o$ converge onto a compact set $E^o$, then those from $\tilde{G}$ converge onto the union $E \cup E^o$, which may be larger than $E$. These two procedures enable us to construct dfs $F^*$ with marginal densities $f_0$ and meta dfs $G^* = F^* \circ K$ with marginal densities $g_0$ which exhibit unexpected behaviour:

(Ex.1) $G^*$ and $G$ have the same asymptotics, but the scaled sample clouds from $F^*$ converge to a Poisson point process which lives on the diagonal. (See Theorem 3.7 and 3.8, and Example 2.)

(Ex.2) The scaled sample clouds from $G^*$ converge onto $E^* = A \cup E_{00}$, where $E_{00}$ is the *diagonal cross*

$$E_{00} = \{r\delta \mid 0 \leq r \leq 1, \ \delta \in \{-1,1\}^d\}, \tag{3.3}$$

and $A \subset [-1,1]^d$ any compact star-shaped set with continuous boundary. The dfs $F^*$ and $F$ have the same asymptotics. The density $f^*$ is asymptotic to $f$ on every ray which does not lie in a coordinate plane. (See Example 2.)

What does the copula say about the asymptotics? Everything, since it determines the df if the marginals are given; nothing, since the examples above show that there is no relation between the asymptotics of $F^*$ and the asymptotics of $G^*$ even with the prescribed marginals $f_0$ and $g_0$. One might hope that at least the parameters $\lambda$ and $\theta$, determined by the marginals, might be preserved in the asymptotics. The point process $N^*$ reflects the parameter $\lambda$ in the marginal intensities $\lambda/|t|^{\lambda+1}$. However, $E^* = E_{\lambda^*,\theta^*}$ may hold for any $\lambda^*$ and $\theta^*$ in $(0,\infty)$ by taking $A = E_{\lambda^*,\theta^*}$ in the second example above.

We now start with the technical details.

The construction procedures discussed above will change an original df $\hat{F}$ with marginals $F_0$ into a df $\tilde{F}$ whose marginals $\tilde{F}_j$ are tail equivalent to $F_0$. This is no serious obstacle.

**Proposition 3.1.** *Let the scaled sample clouds from $\tilde{F}$ converge to a point process $\tilde{N}$, and let the scaled sample clouds from $\tilde{G} = \tilde{F} \circ K$ converge onto the compact set $\tilde{E}$. If the marginals $\tilde{F}_j$ are continuous and strictly increasing and tail asymptotic to $F_0$ then there exists a df $F^*$ with marginals $F_0$ such that $F^*$ has the same asymptotics as $\tilde{F}$ and $G^* = F^* \circ K$ has the same asymptotics as $\tilde{G}$.*

*If moreover $\tilde{F}$ has a density $\tilde{f}$ with marginals asymptotic to $f_0$ in $\pm\infty$, then $F^*$ has a density $f^*$, and for any vector $\mathbf{w}$ with non-zero coordinates and any sequence $\mathbf{w}_n \to \mathbf{w}$ and $r_n \to \infty$ there is a sequence $\mathbf{w}'_n \to \mathbf{w}$ such that $f^*(r_n\mathbf{w}_n) \sim \tilde{f}(r_n\mathbf{w}'_n)$.*

**Proof** Let $F^* = \tilde{F} \circ K_F$ be the meta df based on $\tilde{F}$ with marginals $F_0$. The components $K_{Fj} = \tilde{F}_j \circ F_0$ are homeomorphisms and satisfy $K_{Fj}(t) \sim t$ for $|t| \to \infty$. (Here we use that the marginal tails vary regularly with exponent $-\lambda \neq 0$.) It follows that

$$\|K_F(\mathbf{z}) - \mathbf{z}\|/\|\mathbf{z}\| \to 0 \qquad \|\mathbf{z}\| \to \infty. \tag{3.4}$$

This ensures that $\tilde{F}$ and $F^*$ have the same asymptotics. (For any $\epsilon > 0$ the distance between the scaled sample point $\mathbf{Z}/c_n$ and $K_F(\mathbf{Z})/c_n$ is bounded by $\epsilon\|\mathbf{Z}\|/c_n$ for $\|\mathbf{Z}\| \geq \epsilon c_n$ and $n \geq n_\epsilon$.) A similar argument shows that $\tilde{G} = \tilde{F} \circ K$ and $G^* = F^* \circ K$, the meta df based on $\tilde{G}$ with marginals $G_0$, have the same asymptotics. Here we use that $\tilde{G}_j = \tilde{F}_j \circ K_0$ is tail asymptotic to $G_0 = F_0 \circ K_0$ since $\tilde{F}_j$ is tail asymptotic to $F_0$. Under the assumptions on the density the Jacobean of $K_F$ is asymptotic to one in the points $r_n\mathbf{w}'_n$ and (3.4) gives the limit relation with $r_n\mathbf{w}'_n = K_F^{-1}(r_n\mathbf{w}_n)$. ¶

In general, the densities $f^*$ and $\tilde{f}$ (in the notation of the above proposition) are only weakly asymptotic, as in Proposition 1.8 in [2]. The density $f^*$ on $\mathbf{z}$-space is related to $f$ in the same way as the density $g^*$ is related to $g$. If $f^* \asymp f$ or $f^* \leq Cf$ or $f^* \sim f$, then these relations also holds for $g^*$ and $g$, and vice versa. Similarly for the marginals: $g_j^* \sim g_0$ in $\infty$ implies $f_j^* \sim f_0$ in $\infty$. These results are formalized in the lemma below.

**Lemma 3.2.** *If $F^*$ has density $f^*$ with marginals $f_j^*$ and $G^* = F^* \circ K$ has density $g^*$ with marginals $g_j^*$, then*

$$g^*(\mathbf{x})/g(\mathbf{x}) = f^*(\mathbf{z})/f(\mathbf{z}) \qquad g_j^*(s)/g_0(s) = f_j^*(t)/f_0(t) \qquad \mathbf{z} = K(\mathbf{x}),\ t = K_0(s).$$

**Proof** The Jacobean drops out in the quotients. ¶

## 3.1 Block partitions

We introduce partitions of $\mathbb{R}^d$ into bounded Borel sets $B_n$. In our case the sets $B_n$ are coordinate blocks. Since our dfs have continuous marginals the boundaries of the blocks are null sets, and we shall not bother about boundary points, and treat the blocks as closed sets. To construct such a *block partition* start with an increasing sequence of cubes

$$s_nC = [-s_n, s_n]^d \qquad 0 < s_1 < s_2 < \cdots, \; s_n \to \infty \qquad C = [-1, 1]^d.$$

Subdivide the ring $R_n = s_{n+1}C \setminus s_nC$ between two successive cubes into blocks by a symmetric partition of the interval $[-s_{n+1}, s_{n+1}]$ with division points $\pm s_{nj}$, $j = 1, \ldots, m_n$, with

$$-s_{n+1} < -s_n < \cdots < -s_{n1} < s_{n1} < \cdots < s_{nm_n} = s_n < s_{nm_n+1} = s_{n+1}.$$

This gives a partition of the cube $s_{n+1}C$ into $(2m_n + 1)^d$ blocks of which $(2m_n - 1)^d$ form the cube $s_nC$. The remaining blocks form the ring $R_n$. The meta transformation $K$ transforms block partitions in $\mathbf{x}$-space into block partitions in $\mathbf{z}$-space. A comparison of the original block partition with its transform gives a good indication of the way in which the meta transformation distorts space.

**Definition 5.** *A partition of $\mathbb{R}^d$ into Borel sets $A_n$ is* regular *if the following conditions hold:*

*1) The sets $A_n$ are bounded and have positive volume $|A_n| > 0$;*

*2) Every compact set is covered by a finite number of sets $A_n$;*

*3) The sets $A_n$ are relatively small: There exist points $\mathbf{p}_n \in A_n$ with norm $\|\mathbf{p}_n\| = r_n > 0$ such that for any $\epsilon > 0$, $A_n \subset \mathbf{p}_n + \epsilon r_n B$, $n \geq n_\epsilon$.*

The block partition introduced above is regular if and only if $s_{n+1} \sim s_n$ and $\Delta_n/s_n \to 0$ where $\Delta_n$ is the maximum of $s_{n1}, s_{n2} - s_{n1}, \ldots, s_{nm_n} - s_{nm_n-1}$. Regular partitions give a simple answer to the question: If one replaces $f$ or $g$ by a discrete distribution, how far apart are the atoms allowed to be if one wants to retain the asymptotic behaviour of the sample clouds from the given density?

**Proposition 3.3.** *Let $A_1, A_2, \ldots$ be a regular partition. Suppose the sample clouds from the probability distribution $\mu$ scaled by $r_n$ converge onto the compact set $E$. Let $\tilde{\mu}$ be a probability measure such that $\tilde{\mu}(A_n) = \mu(A_n)$ for $n \geq n_0$. Then the sample clouds from $\tilde{\mu}$ scaled by $r_n$ converge onto $E$.*

**Proof** Let $\mathbf{p} \in E$, and $\epsilon > 0$. Let $\mu_n$ denote the mean measure from the scaled sample cloud from $\mu$ and $\tilde{\mu}_n$ the same for $\tilde{\mu}$. Then $\mu_n(\mathbf{p} + (\epsilon/2)B) \to \infty$. Because the sets $A_n$ are relatively small there exists $n_1$ such that any set $A_n$ which intersects the ball $r_n\mathbf{p} + r_n\epsilon B$ with $n \geq n_1$ has diameter less than $\epsilon r_n/2$. Let $U_n$ be the union of the sets $A_n$ which intersect $r_n\mathbf{p} + (r_n\epsilon/2)B$. Then $U_n \subset r_n\mathbf{p} + \epsilon r_n B$ and hence

$$\mu_n(\mathbf{p} + (\epsilon/2)B) \leq \mu_n(U_n/r_n) = \tilde{\mu}_n(U_n/r_n) \leq \tilde{\mu}_n(\mathbf{p} + \epsilon B).$$

Similarly $\tilde{\mu}_n(U^c) \to 0$ for any open set $U$ which contains $E$. ¶

**Remark 1.** The result also holds if $\tilde{\mu}(A_n) \asymp \mu(A_n)$ provided $\mu(A_n)$ is positive eventually. ◊

There is an analogous result for regular partitions in $\mathbf{z}$-space.

**Proposition 3.4.** *Suppose $\pi \in \mathcal{D}_\lambda(\rho)$ with scaling constants $c_n$. Let $A_1, A_2, \dots$ be a regular partition and let $\tilde{\pi}$ be a probability measure on $\mathbb{R}^d$ such that $\tilde{\pi}(A_n) = \pi(A_n)$ for $n \geq n_0$. Then $\tilde{\pi} \in \mathcal{D}_\lambda(\rho)$ with scaling constants $c_n$.*

**Proof** Any closed block $A \subset \mathbb{R}^d \setminus \{\mathbf{0}\}$ whose boundary carries no $\rho$-mass is contained in an open block $U$ with $\rho(U) < \rho(A) + \epsilon$. As in the proof of the previous proposition for $n \geq n_1$ there is a union $U_n$ of atoms $A_n$ such that $A \subset U_n/c_n \subset U$. ¶

For excess measures $\rho$ with a continuous positive density $h$ there is a converse. For $\pi \in \mathcal{D}_\lambda(\rho)$ with limit $\rho$ there is a regular partition $A_1, A_2, \dots$ such that $\pi(A_n) \sim \int_{A_n} f(\mathbf{z})d\mathbf{z}$ with $f \in \mathcal{F}_\lambda$, and with the same scaling constants. See [4], Theorem 16.27. This result vindicates our use of densities in $\mathcal{F}_\lambda$. Not every distribution in the domain of $h$ has a density, or even a density in $\mathcal{F}_\lambda$, but every such distribution is close to a density in $\mathcal{F}_\lambda$ in terms of a regular partition.

We thus have the following simple situation: $A_1, A_2, \dots$ is a block partition in $\mathbf{x}$-space and $B_1 = K(A_1), B_2 = K(A_2), \dots$ the corresponding block partition in $\mathbf{z}$-space. Let $\tilde{\pi}$ be a probability measure in $\mathbf{z}$-space and $\tilde{\mu}$ a probability measure in $\mathbf{x}$-space, linked by $K$, i.e. $\tilde{\pi} = K(\tilde{\mu})$. Then $\tilde{\pi}(A_n) = \tilde{\mu}(B_n)$ for all $n$. So

$$\tilde{\pi}(A_n) \sim \int_{A_n} f(\mathbf{z})d\mathbf{z} \qquad \Longleftrightarrow \qquad \tilde{\mu}(B_n) \sim \int_{B_n} g(\mathbf{x})d\mathbf{x}. \tag{3.5}$$

**Theorem 3.5.** *If both partitions are regular and one of the equivalent asymptotic equalities in (3.5) holds, then the sample clouds from $\tilde{\pi}$ scaled by $c_n$ converge to the Poisson point process with intensity $h$ in* (2.6)*, and the sample clouds from $\tilde{\mu}$ scaled by $r_n$ converge onto the set $E = E_{\lambda,\theta}$ in* (2.8)*.*

**Proof** Combine Proposition 3.3 and 3.4. ¶

Unfortunately the meta transformation $K$ is very non-linear. Regularity of one block partition does not imply regularity of the other block partition.

We first consider the case when the block partition $(A_n)$ in $\mathbf{x}$-space is regular, but $(B_n)$ is not. The block partition $(A_n)$ in $\mathbf{x}$-space is based on a sequence of cubes $s_nC = [-s_n, s_n]^d$. Successive cubes are of the same size asymptotically, $s_{n+1} \sim s_n$. The cubes $t_nC$ in $\mathbf{z}$-space with $t_n = K_0(s_n)$ may grow very fast. It is possible that $t_n << t_{n+1}$, as in Proposition 3.6 below. The corresponding partition with blocks $B_n = K(A_n)$ in $\mathbf{z}$-space then certainly is not regular.

**Proposition 3.6.** *Let $\epsilon \in (0,1)$. There is a sequence $0 < s_1 < s_2 < \cdots$ such that $s_n \to \infty$ and $s_{n+1} \sim s_n$, and such that*

$$t_n = K_0(s_n) = n^{n^{n^{1-\epsilon}}}.$$

**Proof** We have $g_0 \sim e^{-\psi}$ implies $1 - G_0(s) \sim a(s)g_0(s) \sim e^{-\Psi(s)}$ where $\Psi$ like $\psi$ varies regularly with exponent $\theta$. Write $s_n = e^{\sigma_n}$ and $\tau\Psi(s_n) \sim e^{\theta r(\sigma_n)}$ where $r$ is a $C^2$ function with $r'(t) \to 1$ and $r''(t) \to 0$ as $t \to \infty$, and $\tau := 1/\lambda$. It has been shown in [2] (Equation (1.13)) that

$$K_0(s) = t \sim ce^{\varphi(s)} \quad s \to \infty, \quad \varphi(s) = \tau q(\Psi(s)) \sim \tau\Psi(s),$$

for some positive constant $c$. This gives $\log t_n = \log K_0(s_n) \sim e^{\theta r(\sigma_n)}$. Since $\log\log t_n = n^{1-\epsilon}\log n + \log\log n$ has increments which go to zero, so does $\theta r(\sigma_n)$, and hence also $\sigma_n$ since $r'$ tends to one. It follows that $s_{n+1} \sim s_n$. ¶

Choose $s_{nm_n-1} = s_{n-1}$. Then the cube $[s_{n-1}\mathbf{e}, s_{n+1}\mathbf{e}]$ is a union of $2^d$ blocks in the partition, and so is the cube $[t_{n-1}\mathbf{e}, t_{n+1}\mathbf{e}]$ in $\mathbf{z}$-space; $\mathbf{e} = (1, \dots, 1)$ denotes a vector of ones in $\mathbb{R}^d$. The union $U$ of these latter cubes has the property that the scaled sets $U/t_n$ converge to $(0, \infty)^d$ for $t_n \to \infty$ if $t_n << t_{n+1}$.

**Theorem 3.7.** *Assume the standard set-up with the excess measure $\rho$ of the original distribution which does not charge the coordinate planes. Let $\tilde{\rho}$ be an excess measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ with marginal densities $\lambda/|t|^{\lambda+1}$, and assume that for each orthant $Q_\delta$, $\delta \in \{-1, 1\}^d$, the restrictions of $\tilde{\rho}$ and $\rho$ to $Q_\delta$ have the same univariate marginals. One may choose $\tilde{F}$ such that its marginals are tail asymptotic to $F_0$, such that the sample clouds converge to the Poisson point process $\tilde{N}$ with mean measure $\tilde{\rho}$, and such that the sample clouds from the df $\tilde{G} = \tilde{F} \circ K$ converge onto the limit set $E_{\lambda,\theta}$ in* (2.8).

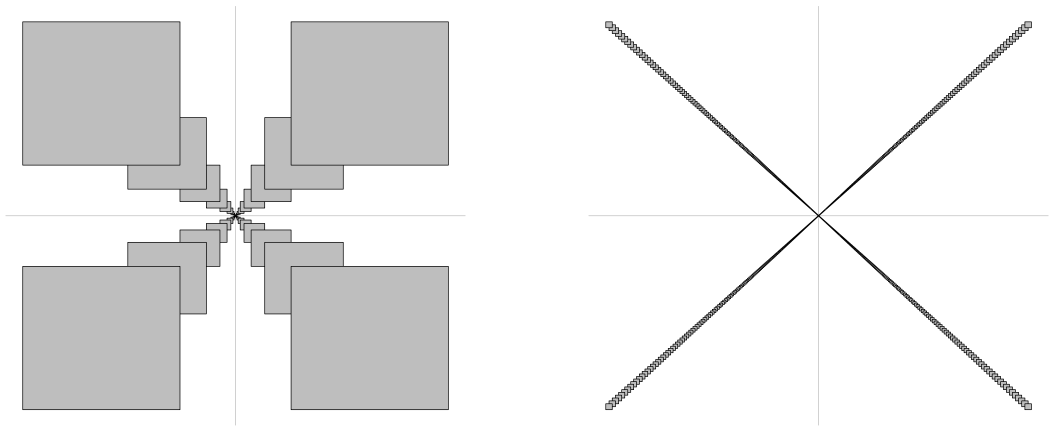
**Proof** We sketch the construction. Choose $\hat{F} \in \mathcal{D}_\lambda(\tilde{\rho})$ with density $\hat{f}$ such that the sample clouds from $\hat{F}$ scaled by $c_n$ converge to $\tilde{N}$. For $\delta \in \{-1, 1\}^d$ let $U_\delta$ be the image of the union $U$ in $Q_\delta$ by reflecting coordinates for which $\delta_j = -1$ (see Figure 1 for an illustration). Let $\tilde{f}$ agree with $\hat{f}$ on the $2^d$ sets $U_\delta$ and with $f$ elsewhere, so that, by the remark above on the convergence of the scaled sets $U$, $\tilde{f}$ and $\hat{f}$ differ only on an asymptotically negligible set. Alter $\tilde{f}$ on a bounded set to make it a probability density. Then the sample clouds from $\tilde{F}$ scaled by $c_n$ converge to $\tilde{N}$. In the corresponding partition $(A_n)$ on $\mathbf{x}$-space we only change the measure on the "tiny" blocks $[s_{n-1}, s_{n+1}]^d$ (with $s_{n-1} \sim s_{n+1}$) around the positive diagonal, and their reflections. Hence the scaled sample clouds from $\tilde{G} = \tilde{F} \circ K$ converge onto $E_{\lambda,\theta}$. ¶

We now discuss the second case: the block partition $(B_n)$ in $\mathbf{z}$-space is regular, but $(A_n)$ is not. As before, a block partition on $\mathbf{x}$-space is determined by an increasing sequence of cubes $s_nC = [-s_n, s_n]^d$, and for each $n$ a symmetric partition of $[-s_n, s_n]$ given by a finite sequence of points $s_{n1} < \cdots < s_{nm_n} = s_n$ in $(0, s_{n+1})$. The image blocks are determined by $t_n = K_0(s_n)$ and $t_{nj} = K_0(s_{nj})$. This makes it convenient to define these quantities in terms of upper quantiles. Choose $m_n = n$, probabilities $p_n = e^{-\sqrt{n}}$, and write

$$1 - F_0(t_n) = 1 - G_0(s_n) = p_n \qquad 1 - F_0(t_{nj}) = 1 - G_0(s_{nj}) = np_n/j. \tag{3.6}$$

One may see $t_{nj}$ as a function $T$ of $j/np_n$, and so too $s_{nj}$ as a function $S$ of $\log(j/np_n)$:

$$t_{nj} = T(je^{\sqrt{n}}/n) \qquad s_{nj} = S(\sqrt{n} + \log j - \log n) \qquad j = 1, \dots, n, \ n = 1, 2, \dots$$



(a) **z**-space

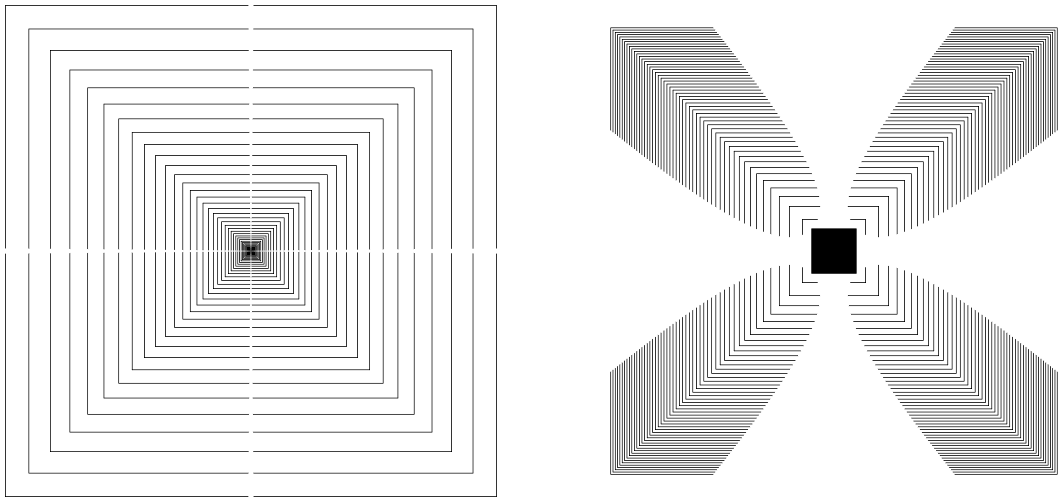
(b) **x**-space (regular)

Figure 1: Panel (a): Schematic representation of the four sets $U_\delta = \bigcup_n [\delta_1 t_{n-1}\mathbf{e}, \delta_2 t_{n+1}\mathbf{e}]$ in $\mathbb{R}^2$ for $\delta = (\delta_1, \delta_2) \in \{-1, 1\}^2$; see Theorem 3.7. Panel (b): the image of the sets $U_\delta$ in **x**-space. The division points of the partitions $(A_n)$ and $(B_n)$ are $s_n = n^{\sqrt{n}} \log n$ and $t_n = e^{s_n} = n^{n^{\sqrt{n}}}$ respectively. Use Proposition 3.6 with $K_0(s) = e^s$, $s > s_0$ and $\epsilon = 0.5$.

The increasing functions $T$ and $S$ vary regularly with exponents $1/\lambda$ and $1/\theta$ since the inverse functions to $1 - F_0$ and $-\log(1 - G_0)$ vary regularly in zero with exponents $-1/\lambda$ and $1/\theta$. It follows that $T((j_n/n)e^{\sqrt{n}})/T(e^{\sqrt{n}}) \to u^{1/\lambda}$ if $j_n/n \to u \in [0, 1]$, and hence $t_{n1}/t_n \to 0$ and the maximal increment $t_{nj} - t_{nj-1}$, $j = 2, \ldots, n$, is $o(t_n)$. So the block partition in **z**-space is regular. However $s_{n1} \sim s_n$ since $\log n = o(\sqrt{n})$ implies $S(\sqrt{n} - \log n)/S(\sqrt{n}) \to 1$. Figure 2 depicts sequences of cubes $s_n C$ and $t_n C$ in $\mathbb{R}^2$ on which partitions $(A_n)$ and $(B_n)$ are based in the special case when $s_n = \sqrt{n}$ and $t_n = K_0(s_n) = e^{s_n} = e^{\sqrt{n}}$, along with subintervals $[-e^{\sqrt{n}}/n, e^{\sqrt{n}}/n] \times \{e^n\}$ in **z**-space mapping onto $[-\sqrt{n} + \log n, \sqrt{n} - \log n] \times \{\sqrt{n}\}$ in **x**-space, which correspond to the partition blocks intersecting the coordinate axes.

**Theorem 3.8.** *Assume the standard set-up. There exists a df $\tilde{F}$ such that the original df $F$ and $\tilde{F}$ have the same asymptotics, and the scaled sample clouds from the corresponding meta df $\tilde{G}$ converge onto the diagonal cross $E_{00}$ in* (3.3).

**Proof** Let $(B_n)$ be the regular block partition in **z**-space as above. Construct a density $\tilde{f}$ by deleting the mass of the original df $F$ in the blocks $B_n$ which intersect one of the $d$ coordinate planes, except for the block containing the origin, where we increase the density by a factor $c > 1$ to compensate for the loss of mass elsewhere. The new density $\tilde{f}$ agrees with $f$ on every block which does not intersect a coordinate plane. The relation $t_{n1}/t_n \to 0$ implies that $\tilde{f}$ and $f$ agree outside a vanishing neighborhood around the coordinate planes. On the other hand the relation $s_{n1} \sim s_n$ implies that $\tilde{g}$ and $g$ only agree



(a) **z**-space (regular) (b) **x**-space

Figure 2: Panel (a): blocks $[-t_n, t_n]^2 = [-e^{\sqrt{n}}, e^{\sqrt{n}}]^2$, $n \in \{2, 4, 6, \dots, 100\}$ in **z**-space with the sets $[-e^{\sqrt{n}}/n, e^{\sqrt{n}}/n] \times [e^{\sqrt{n-1}}, e^{\sqrt{n}}]$ and their reflections, which correspond to the blocks of partition $(B_n)$ intersecting the axes, deleted. Panel (b): image of the blocks $([-\sqrt{n}, \sqrt{n}]^2)$ and deleted blocks $([-\sqrt{n} + \log n, \sqrt{n} - \log n] \times [\sqrt{n-1}, \sqrt{n}])$ from Panel (b) in **x**-space under the meta transformation $K_0(s) = e^s$, $s > s_0$.

on a vanishing neighborhood around the diagonals. Thus $\tilde{f}(r_n \mathbf{w}_n) = f(r_n \mathbf{w}_n)$ eventually for $r_n \to \infty$ if $\mathbf{w}_n$ converges to a vector $\mathbf{w}$ with non-zero components. Conversely, if $r_n \to \infty$ and $\mathbf{w}_n$ converges to a vector $\mathbf{w}$ which does not lie on one of the $2^d$ diagonal rays, then $\tilde{g}(r_n \mathbf{w}_n) = 0$ eventually.

The function $\tilde{f}$ satisfies the same limit relation $\tilde{f}(r_n \mathbf{w}_n)/f(r_n \mathbf{e}) \to h(\mathbf{w})$ for $r_n \to \infty$ as $f$, provided $\mathbf{w} = \lim_{n\to\infty} \mathbf{w}_n$ has non-zero coordinates. Dominated convergence, by the inequality $\tilde{f} \le f$ outside $t_1 C$, gives $\mathbf{L}^1$-convergence outside centered balls. It follows that $\tilde{F}$ and $F$ have the same asymptotics. The scaled sample clouds from $\tilde{g}$ converge onto the diagonal cross $E_{00}$. ¶

The incompatibility of the partitions $(A_n)$ and $(B_n) = (K(A_n))$ introduced in this section gives one technical explanation for the peculiar sensitivity of the limit shape for the meta distribution. If we regard the atoms of the partition $(B_n)$ as nerve cells, then regularity of $(A_n)$ will make the region around the coordinate planes in **z**-space far more sensitive than the remainder of the space, and it is not surprising that cutting away these regions has drastic effects on the limit.

## 3.2 Mixtures

For a wide class of star-shaped sets $A \subset [-1, 1]^d$ it is possible to alter the original density so that the limit set $E$ of the new meta density is the union $A \cup E_{00}$, where $E_{00}$ is the diagonal cross in (3.3).

**Theorem 3.9.** *Assume the standard set-up. Let $A$ be a star-shaped closed subset of the unit cube $[-1,1]^d$ with a continuous boundary and containing the origin as interior point, and let $E_{00}$ be the diagonal cross in (3.3). There exists a df $\tilde{F}$ with the same asymptotics as $F$, such that the scaled sample clouds from the meta distribution $\tilde{G}$ converge onto the set $E = A \cup E_{00}$.*

**Proof** Let $\hat{G} = \hat{F} \circ K$ where $\hat{F}$ has marginal densities $f_0$, and let $G^o = F^o \circ K$ where $F^o$ has continuous marginals $F_j^o$ with lighter tails than $F_0$, see (3.2). Let $d\tilde{F}$ agree with $d(\hat{F} + F^o)$ outside a bounded set. The sample clouds from $F^o$ scaled by $c_n$ converge onto $\{\mathbf{0}\}$ since $n(1 - F_j^o(\epsilon c_n)) + nF_j^o(-\epsilon c_n) \to 0$ as $n \to \infty$, $j = 1, \dots, d$, $\epsilon > 0$. So $\tilde{F}$ and $\hat{F}$ have the same asymptotics.

Let $n_A$ denote the gauge function of $A$ (or its interior), and let $g^o(\mathbf{x}) = g_*(n_A(\mathbf{x}))$ for a continuous decreasing positive function $g_*$ on $[0, \infty)$. We assume that $g_*$ varies rapidly. The function $g^o$ is continuous and $0 < g^o(\mathbf{x}) \le \bar{g}(\mathbf{x})$. Let $d\tilde{G}$ agree with $d\hat{G}(\mathbf{x}) + g^o(\mathbf{x})d\mathbf{x}$ outside a bounded set. We may assume that $d\hat{G}$ does not charge coordinate planes $\{x_j = c\}$ and charges all coordinate slices $\{c_1 < x_j < c_2\}$. Then the marginals are continuous and strictly increasing.

Choosing $\hat{F}$ to have the same asymptotics as the original df $F$, and so that the scaled sample clouds from $\hat{G}$ converge onto the diagonal cross $E_{00}$ (see Theorem 3.8 and Proposition 3.1), we obtain $\tilde{F}$ and $F$ with the same asymptotics, and convergence of the scaled sample clouds from $\tilde{G}$ onto $E_{00} \cup A$. ¶

**Example 2.** Let $\bar{g}$ be a density with cubic level sets: $\bar{g}(\mathbf{x}) = g_*(\|\mathbf{x}\|_\infty)$ with $g_*$ as in the above theorem. The marginal densities $\bar{g}_0$ are symmetric and equal, and asymptotic to $(2s)^{d-1}g_*(s)$. Let $\bar{g}_0(\bar{b}_n) \sim 1/n$. We may choose $g_*$ such that $\bar{b}_n \sim b_n$ and $n\bar{g}_0(b_n) \to 0$, (see Lemma A.1 and Proposition A.2 for details). It follows that the sample clouds from $\bar{G}$ scaled by $\bar{b}_n$ converge onto the cube $[-1,1]^d$, hence also the sample clouds scaled by $b_n$.

In the above theorem, take $\hat{F} = F$ and $G^o = \bar{G}$. Then $\tilde{F}$ has the same asymptotics as $F$ and the scaled sample clouds from $\tilde{G} = \tilde{F} \circ K$ converge onto the cube $[-1,1]^d$.

If we choose the measure $d\hat{F}$ to be the image of the marginal $dF_0$ under the map $t \mapsto t\mathbf{e}$, then $d\hat{F}$ lives on the diagonal. The scaled sample clouds from $\tilde{F}$ converge to the Poisson point process on the diagonal with intensity $\lambda/|t|^{\lambda+1}$ in the parametrization above, and the scaled sample clouds from $\tilde{G}$ converge onto $A \cup E_{00}$. If we choose $A = E_{\lambda,\theta}$ then the dfs $G$ and $\tilde{G}$ have the same asymptotics, but we may also choose $A = E_{\lambda^*,\theta^*}$ for other values $\lambda^*$ and $\theta^*$ in $(0, \infty)$. ◇

We have shown that slight changes in $F$, changes which do not affect the asymptotics or the marginals, may yield a meta distribution $\tilde{G}$ with the marginals of $G$ but with different asymptotics. This makes it possible to start out with a Poisson point process $N^*$ and a star-shaped set $E^*$, and construct dfs $F^*$ and $G^*$ with marginal densities $f_0$ and $g_0$ such that the sample clouds from $F^*$ converge to $N^*$ and those from $G^*$ converge onto $E^*$. The only condition is that the mean measure $\rho^*$ of $N^*$ is an excess measure with marginal densities $\lambda/|t|^{\lambda+1}$ and that $E^*$ is a closed star-shaped subset of $[-1,1]^d$ containing the $2^d$

vertices and having a continuous boundary.

Although the shape of the limit set is rather unstable under even slight perturbations of the original distribution, one may note the persistence of the diagonal cross as a subset of the limit set. Due to equality and symmetry of the marginals, clearly the points on the $2^{d-1}$ diagonals in $\mathbf{z}$-space are mapped into the points on the diagonals in $\mathbf{x}$-space. However, Figure 1 shows that much larger subsets of the open orthants in $\mathbf{z}$-space are mapped on the neighborhood of the diagonals in $\mathbf{x}$-space.

**Proposition 3.10.** *Consider the standard set-up with the original df $F$ having equal marginals. Let $\tilde{F}$ be a df on $\mathbb{R}^d$ whose marginals are equal and symmetric and tail asymptotic to those of $F$. Assume that the sample clouds from $\tilde{F}$ converge to a point process with mean measure $\tilde{\rho}$, where $\tilde{\rho}$ charges $(0,\infty)^d$. If sample clouds from the meta distribution $\tilde{G} = \tilde{F} \circ K$ can be scaled to converge onto a limit set $\tilde{E}$ then $\tilde{E}$ contains the vertex $\mathbf{e} = (1,\ldots,1)$.*

**Remark 2.** Limit sets are always star-shaped (see Proposition 4.1 in [7]). Hence, if the limit set $\tilde{E}$ contains $\mathbf{e}$, it also contains the line segment $E_{00}^+ = E_{00} \cap (0,\infty)^d$ joining $\mathbf{e}$ and the origin.

**Proof** To see this we again make use of the block partitions of Section 3.1; in particular consider the situation sketched in Figure 1. Due to symmetry, it suffices to restrict attention to the positive orthant. Consider cubes $C_n^A := [s_n - Ma(s_n), s_n + Ma(s_n)]^d$ centered at diagonal points $s_n\mathbf{e}$ for some $M > 0$, where $a(s)$ is the scale function of the marginal df $G_0$. Recall that $a'(s) \to 0$ and hence $a(s)/s \to 0$ as $s \to \infty$. These cubes are asymptotically negligible as $C_n^A/s_n \to \{\mathbf{e}\}$ for $s_n \to \infty$. The corresponding cubes in $\mathbf{z}$-space are centered at the diagonal points $t_n\mathbf{e}$ with $t_n = K_0(s_n)$ and given by $C_n^B := K(C_n^A) = [K_0(s_n - Ma(s_n)), K_0(s_n + Ma(s_n))]^d =: [t_{n-1}, t_{n+1}]^d$. The von Mises condition on $1 - G_0$ with scale function $a(s)$ and regular variation of $(1-F_0)^{-1}$ in zero with exponent $-1/\lambda$ give

$$\begin{aligned}
\lim_{n\to\infty} \frac{t_{n-1}}{t_n} &= \lim_{s_n\to\infty} \frac{K_0(s_n - Ma(s_n))}{K_0(s_n)} = \lim_{s_n\to\infty} \frac{(1-F_0)^{-1}(1-G_0(s_n - Ma(s_n)))}{(1-F_0)^{-1}(1-G_0(s_n))} \\
&= \lim_{s_n\to\infty} \Big(\frac{1-G_0(s_n)}{1-G_0(s_n - Ma(s_n))}\Big)^{1/\lambda} \frac{L(1-G_0(s_n - Ma(s_n)))}{L(1-G_0(s_n))} \\
&= \lim_{s_n\to\infty} \Big(\frac{e^{-M}(1-G_0(s_n))}{1-G_0(s_n)}\Big)^{1/\lambda} \frac{L(1-G_0(s_n))}{L(e^{-M}(1-G_0(s_n)))} = e^{-M/\lambda},
\end{aligned}$$

for a slowly varying function $L$. Similarly, $t_{n+1}/t_n \to e^{M/\lambda}$, and thus $C_n^B/t_n \to [e^{-M/\lambda}, e^{M/\lambda}]^d$. Note that for $M$ large, this limit constitutes a large subset of $(0,\infty)^d$ in $\mathbf{z}$-space in that it will eventually, for $M$ large enough, contain any compact subset of $(0,\infty)^d$.

The points in $C_n^B$ are mapped by the meta transformation $K$ onto the points in $C_n^A$, thus preserving the mass $\mathbb{P}\{\mathbf{Z} \in C_n^B\} = \mathbb{P}\{\mathbf{X} \in C_n^A\}$. This shows that most of the mass on $(0,\infty)^d$ in $\mathbf{z}$-space is concentrated on just a neighborhood of $E_{00}^+$ in $\mathbf{x}$-space, and the assumption on $\tilde{\rho}$ ensures that the limit of the scaled sample clouds contain the vertex $\mathbf{e}$. ¶

## 3.3 Extremes and high risk scenarios

There are different ways in which one can look at multivariate extremes. The focus of the paper so far has been on a global view of sample clouds. We have seen that the asymptotic behaviour of sample clouds is described by a Poisson point process for heavy-tailed distributions, and by the limit set for light-tailed distributions. We would now like to complement these global pictures by looking more closely at the edge of sample clouds. In particular, we discuss asymptotic behaviour of coordinatewise maxima and of exceedances over hyperplanes, termed *high risk scenarios* in [4].

### 3.3.1 Biregular partition and additional notation

We first introduce a *biregular* block partition, a partition which is regular in both **z**-space and in **x**-space. For simplicity, we shall restrict attention to a bivariate situation. Recall the block partition defined in terms of quantiles in (3.6), which was regular in **z**-space, but not in **x**-space. We now refine it to make it biregular. A typical block intersecting the positive horizontal axis in **x**-space has the form

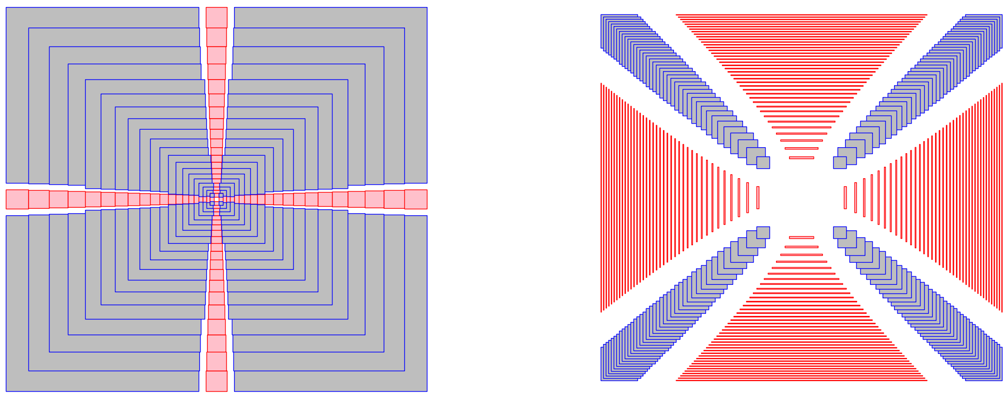
$$[s_{n-1}, s_n] \times [-s_{n1}, s_{n1}] \qquad s_{n-1} \sim s_n \sim s_{n1}. \tag{3.7}$$

It is an elongated thin vertical rectangle almost stretching from one diagonal arm to the other. We subdivide it in the vertical direction into $2n$ congruent rectangles by adding the division points

$$-(n-1)s_{n1}/n, \ldots, -s_{n1}/n, 0, s_{n1}/n, \ldots, (n-1)s_{n1}/n.$$

The corresponding partition $(A_n)$ in **x**-space is regular, and so is the partition $(B_n)$ with $B_n = K(A_n)$ in **z**-space, since it is a refinement of a regular partition.

We shall be interested in three sets $C$, $D$, and $O$ which are disjoint and fill $\mathbb{R}^2$. They are defined in terms of the biregular partition we have introduced above. Roughly speaking, $C$ consists of the blocks around the coordinate axes, $D$ consists of the blocks around the diagonals, and $O$ is the remainder. Due to the symmetry of the partition, we distinguish four subsets of $C$ associated with the four halfaxes: $C_N$ with $\{0\} \times (0, \infty)$, $C_E$ with $(0, \infty) \times \{0\}$, and their negative counterparts $C_S$ and $C_W$. Similarly, $D_{NE}$ is the restriction of $D$ to $(0, \infty)^2$ and analogously for the other quadrants $D_{SE}$, $D_{SW}$ and $D_{NW}$ clockwise. Here is the assignment of the blocks $A_n$ (or $B_n$) to the sets $C$, $D$ and $O$: In the set $C_E$ we put the rectangles in (3.7), for $n \geq n_0$, or rather the $2n$ blocks into which these rectangles have been subdivided. Similarly, for $C_N$, $C_W$ and $C_S$. In the diagonal sets $D$ we put all blocks in the ring $R_n$ between successive squares, which are determined by subdivision points $\pm s_{nj}$ with $j > j_n = [\sqrt{n}]$. We claim that $D_{NE}$ asymptotically fills up the open positive quadrant in **z**-space. For this we have to show that $t_{nj_n}/t_n \to 0$. By definition $t_{nj}$ is the upper $q$-quantile for the df $F_0$ for $q = ne^{-\sqrt{n}}/j$. Let $t_n$ be the upper quantile for $p_n = e^{-\sqrt{n}}$ and $r_n$ the upper quantile for $q_n = \sqrt{n}e^{-\sqrt{n}}$. We claim that $r_n/t_n \to 0$. This follows by regular variation of the tail of $F_0$ because $p_n/q_n \to 0$. So in **z**-space the diagonal set $D$



(a) **z**-space (b) **x**-space

Figure 3: Panel (a): Sequences of rectangles $[t_{n-1}, t_n] \times [\overline{t_n}, t_n]$ filling region $C_E$ and of cubes $[\overline{t_n}, t_n]^2$ filling region $D_{NE}$, and their reflections in other quadrants with $t_n = e^{\sqrt{n}}$ and $\overline{t_n} = (\lceil\sqrt{n}\rceil/n)t_n$ for $n \in \{1, 2, \dots, 20\}$. Panel (b): The images of the sets in (a) in **x**-space under the coordinatewise map $t = K_0(s) = e^s$ for $n \in \{10, 15, \dots, 200\}$.

asymptotically fills up the whole plane apart from the coordinate axes; in **x**-space the coordinate set $C$ fills up the whole plane apart from the two diagonal lines; and there is still a lot of space left for the set $O$ (note that $t_{n1}/t_{nj_n} \to 0$). See Figure 3 for illustration.

The asymptotic behaviour of the probability distribution $F$ is known once we know the probability mass $p_n$ of the atomic blocks $A_n$ in **x**-space (or of $B_n = K(A_n)$ in **z**-space) (see Propositions 3.3 and 3.4). The specification of the probability masses $p_n$ of the atoms is not a very efficient way of describing the original distribution and the meta distribution. Instead we shall describe the asymptotic behaviour of $dF$ on the diagonal sector $D_{NE}$, and the asymptotic behaviour of the meta distribution $dG$ on the coordinate sector $C_E$. We first consider the two pure cases, where all mass lives on one of the sets $C$ or $D$.

### 3.3.2 The region $D$

The situation on the region $D$ is similar to that discussed in Theorem 3.8. In **z**-space, let $F^*$ be a df in the domain of an excess measure $\rho$ on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, where $\rho$ has marginal intensities $\lambda/t^{\lambda+1}$, $\lambda > 0$, and no mass on the axes. The measure $\rho$ might be concentrated on one of the two diagonals. Since we are concentrating on the sector $D_{NE}$ we shall assume that $\rho$ charges the positive quadrant. Delete the mass outside $D$, and compensate by adding some mass in a compact set. The marginals of the new df $F$ are still tail asymptotic to $F_0$. The measures $ndF$, scaled properly, converge to $\rho$ weakly on the complement

of centered balls. The sample clouds and the coordinatewise maxima converge under the same scaling. The exponent measure of the max-stable limit distribution is the image $\rho^+$ of the measure $\rho$ under the map $\mathbf{z} = (x, y) \mapsto (x \vee 0, y \vee 0) = \mathbf{z}^+$.

The following proposition describes the relation between the exponent measures for an original df and the associated meta df. It is in line with e.g. Proposition 5.10 in [10]. One can determine the max-stable limit for a meta df from the max-stable limit for the original df and the marginals of the meta distribution. We shall refer to this fact as the *invariance principle* (for coordinatewise extremes) since the max-stable limit distributions have the same copula. For a df $F$ in the domain of attraction of an extreme value limit law with exponent measure $\rho$, we use the notation $F \in DA(\rho)$.

**Proposition 3.11.** *Let $F \in DA(\rho^+)$ be a continuous df on $\mathbb{R}^2$ whose exponent measure $\rho^+$ has marginal intensities all equal $\lambda/t^{\lambda+1}$, $\lambda > 0$. Let $G_0$ be a continuous symmetric df on $\mathbb{R}$ whose tails are asymptotic to a von Mises function. If $G$ is the meta df based on $F$ with marginals equal to $G_0$, then $G \in DA(\sigma^+)$, where $\sigma^+$ and $\rho^+$ are related by a coordinatewise exponential transformation, $\rho^+ = K(\sigma^+)$ with equal components*

$$K : \mathbf{u} \mapsto \mathbf{w}, \qquad w_i = K_0(u_i) = e^{u_i/\lambda} \qquad i = 1, 2.$$

**Proof** The map $K$ is the limit of coordinatewise transformations mapping normalized coordinatewise maxima from $G$ into normalized coordinatewise maxima from $F$, hence it is also a coordinatewise transformation. The von Mises condition on $G_0$ implies that $G_0$ is in the domain of attraction of the Gumbel limit law $\exp\{-e^{-x}\}$, and thus $\sigma^+$ has standard exponential marginals. The relations

$$\rho_i^+[t, \infty) = \sigma_i^+[s, \infty) \iff t^{-\lambda} = e^{-s} \iff t = K_0(s) = e^{s/\lambda}, \quad i = 1, 2,$$

for the marginals determine $K$. ¶

In $\mathbf{x}$-space $D_{NE}$ is a thin strip along the diagonal. However, it follows from Proposition 3.11, that the coordinatewise maxima in $\mathbf{x}$-space, centered and scaled, converge to a max-stable distribution whose exponent measure $\sigma^+$ is the coordinatewise logarithmic transform of $\rho^+$. The measure $\sigma^+$ lives on $[-\infty, \infty)^2 \setminus \{(-\infty, -\infty)\}$. The restriction $\sigma$ to $\mathbb{R}^2$ is the image of the restriction of $\rho^+$ (or of $\rho$) to $(0, \infty)^2$.

Let us now look at the asymptotic behaviour of the high risk scenarios and sample clouds for $F$ and $G$. Let $H_n = \{\xi_n \geq c_n\}$ be halfplanes with direction $\xi_n = (a_n, b_n)$ of norm one, and $c_n \to \infty$. For the heavy-tailed df $F \in \mathcal{D}(\rho)$, the domain of attraction of the excess measure $\rho$, the situation is simple.

**Proposition 3.12.** *If $\xi_n \to \xi$ and $c_n \to \infty$ then $\mathbf{Z}^{H_n} \Rightarrow \mathbf{W}$, where $\mathbf{W}$ has distribution $d\rho_\xi = 1_H d\rho/\rho H$ for $H = \{\xi \geq 1\}$.*

**Proof** The map $(u, v) \mapsto (x, y) = (c_n u, c_n v)$ maps $J_n = \{\xi_n \geq 1\}$ onto $H_n$. By assumption the probability measures $dF$ scaled by $c_n$ and multiplied by a suitable factor converges to $\rho$ weakly on the complement

of centered balls, hence on $H$. This gives the result when $\xi_n = \xi$ for all $n$. For the general case, $\xi_n \to \xi$ use the continuity theorem [4], Proposition 5.13. ¶

The same result holds for the light-tailed meta distribution $G$ provided the limit direction $\xi$ does not lie on one of the axes.

**Proposition 3.13.** *If $\xi_n \to \xi \in (0,\infty)^2$ and $c_n \to \infty$ then $\alpha_n^{-1}(\mathbf{X}^{H_n}) = (\mathbf{X}^{H_n} - (b_n, b_n))/a_n \Rightarrow \mathbf{U}$ where $(b_n, b_n) \in \partial H_n$ and $a_n = a(b_n)$ for the scale function $a$ associated with the marginal density $g_0$. The limit $\mathbf{U}$ has distribution $d\sigma_\xi = 1_H d\sigma/\sigma H$ for $H = \{\xi \geq 0\}$.*

**Proof** This follows from the weak convergence of $d\pi = dG$ normalized by $\alpha_n^{-1}$ and multiplied by $1/p_n$:

$$\alpha_n^{-1}(\pi)/p_n \to \sigma^+/a_0 \quad \text{weakly on } \mathbb{R}^2 \setminus [-\infty, c]^2 \qquad c \in \mathbb{R}$$

for $p_n = \pi(\mathbb{R}^2 \setminus [b_n, \infty)^2)$ and $a_0 = \sigma^+([-\infty,\infty)^2 \setminus [-\infty, 0]^2)$. ¶

Convergence of high risk scenarios implies convergence of sample clouds with the same normalizations for halfplanes that satisfy $\mathbb{P}\{\mathbf{Z} \in H_n\} \sim 1/n$ as in [4], Section 14. For the heavy-tailed df $F$ the convergence of the sample clouds $N_n \Rightarrow N$ weakly on $\{\xi \geq c\}$ for all $c > 0$ is no surprise since $N_n \Rightarrow N$ holds weakly on the complement of centered disks. For the light-tailed df $G$ weak convergence on $\{\xi \geq c\}$ for $\xi \in (0,\infty)^2$ follows from weak convergence on $[-\infty,\infty)^2 \setminus [-\infty, c]^2$. Convergence for horizontal halfspaces $H_n = \{y \geq c_n\}$ presents a different picture, since we do not allow mass to drift off to the vertical line in $-\infty$.

**Proposition 3.14.** *Set $H_t = \{y \geq t\}$ and $\alpha_t(u,v) = (tu, t + a(t)v)$. Then $\alpha_t^{-1}(\mathbf{X}^{H_t}) \Rightarrow (U,V)$ where $U$ and $V$ are independent, $V$ is standard exponential, and $U$ assumes only two values, $\mathbb{P}\{U = -1\} = p_- = c_-/c$ and $\mathbb{P}\{U = 1\} = p_+ = c_+/c$. Here $c_- = \sigma^+(\{-\infty\} \times (0,\infty)) = \rho^+(\{0\} \times (1,\infty)) = \rho((-\infty, 0) \times (1,\infty))$, $c_+ = \sigma(\mathbb{R} \times (0,\infty)) = \rho((0,\infty) \times (1,\infty))$, and $c = \rho(\mathbb{R} \times (1,\infty))$ since by assumption $\rho$ does not charge the axes. If $\mathbb{P}\{\mathbf{X} \in H_{r_n}\} \sim 1/n$ then the sample clouds converge $\tilde{M}_n = \{\alpha_{t_n}^{-1}(\mathbf{X}_1), \ldots, \alpha_{t_n}^{-1}(\mathbf{X}_n)\} \Rightarrow \tilde{M}$ weakly on $\{v \geq c\}$, $c \in \mathbb{R}$. The Poisson point process $\tilde{M}$ lives on two vertical lines, on $\{u = -1\}$ with intensity $p_- e^{-v}$ and on $\{u = 1\}$ with intensity $p_+ e^{-v}$.*

**Proof** It suffices to prove the second relation. The limit point process $M$ with mean measure $\sigma$ above the line $\{v = -C\}$ for $C > 1$ corresponds to the sample points $\mathbf{X}_1, \ldots, \mathbf{X}_n$ above the line $\{y = r_n - Ca(r_n)\}$. The corresponding points scaled by $r_n$ converge to the vertex $(1,1)$ of the limit set $E$. Under the normalization $\alpha_{t_n}^{-1}$ the horizontal coordinate converges to 1, and hence the whole sample cloud converges to the projection of $M$ onto the line $\{u = 1\}$. A similar argument holds for the points of the point process with mean measure $\tilde{\sigma}$ on $\mathbb{R}^2$ associated with restriction of $\rho$ to the quadrant $(-\infty, 0) \times (0,\infty)$. These yield the points on the vertical line through $(-1, 0)$. Since $\rho$ does not charge the vertical axis, this accounts for all points. ¶

On the region $D$ there is a simple relation between the Poisson point processes associated with the exponent measures and the Poisson point processes associated with the high risk scenarios.

#### 3.3.3 The region $C$

We shall consider the counterpart of the class $\mathcal{F}_\lambda$ for light-tailed densities $g$ on the plane. The density $g$ is assumed to be unimodal, continuous and to have level sets $\{g > c\}$ which are scaled copies of a bounded open star-shaped set $S$ in the plane with continuous boundary. Such densities are called homothetic. They have the form $g = g_*(n_S)$ where $g_*$ is a continuous decreasing function on $[0, \infty)$, the density generator, and $n_S$ is the gauge function of the set $S$ (see Table A.3). We shall assume that the density generator is asymptotic to a von Mises function, and that $S$ is a subset of the square $(-1, 1)^2$.

If $S$ is the open unit disk, or more generally a rotund set (i.e., convex with a $C^2$ boundary having positive curvature in each point), then the high risk limit scenarios exist for all directions, and are Gauss-exponential. The associated sample clouds in these directions may be normalized to converge to a Poisson point process with Gauss-exponential intensity $e^{-u^2/2}e^{-v}/\sqrt{2\pi}$ on $\mathbb{R}^2$ for appropriate coordinates $u, v$. See Sections 9-11 in [4].

If the set $S$ is a convex polygon with one boundary point $\mathbf{q} = (q_1, 1)$ on the line $\{y = 1\}$ with $|q_1| < 1$, then the horizontal high risk scenarios converge. The associated excess measure has density $h$, where $h$ has conic level sets $\{h > e^{-t}\} = C + t\mathbf{q}$ for an open cone $C$ in the lower halfplane. The cone $C$ describes the asymptotic behaviour of the set $S - \mathbf{q}$ in the origin.

If $S$ is the square $(-1, 1)$ then the horizontal high risk scenarios, normalized by $\alpha_t(u, v) = (tu, t+a(t)v)$ converge. The associated excess measure has density $1_{[-1,1]}(u)e^{-v}$, and vanishes outside the vertical strip $[-1, 1] \times \mathbb{R}$.

Our density $g$ is defined in terms of a bounded star-shaped set $S$ and the density generator $g_*$ which describes the behaviour of $g$ along rays up to a scale factor depending on the direction. On the other hand we want light-tailed densities with prescribed marginals equal to $g_0$. In general it is not clear whether for a given set $S$ there exists a density generator $g_*$ which produces the marginals $g_0$.

**Example 3.** Let $g_0$ be the standard normal density. Let $g = g_*(n_S)$. Given a bounded open convex set $S$ can one find a density generator $g_*$ such that $g$ has marginals $g_0$? If $S$ is the unit disk, or an ellipse symmetric around the diagonal, then $g_*(r) = ce^{-r^2/2}$ will do. If $S$ is the square $(-1, 1)^2$, then one may choose $g_*(r) \sim ce^{-r^2/2}/r$ so that the tails of the marginals agree with $g_0$, see Section A.2 in [2]. By altering $g$ on a square $[-M, M]^2$ one may achieve standard normal marginals. If $S$ is the diamond spanned by the unit vectors on the four halfaxes, then one can choose $g_*(r) \sim cre^{-r^2/2}$ so that the tails of the marginals agree with $g_0$, see [9]. These three density generators are different. It is not clear how to combine the asymptotic behaviour of these examples.

Let us take a mixture of these three densities with weights 1/3 each. Set $\alpha_t(u, v) = (tu, t+a(t)v)$ where $a(t) = 1/t$ is the scale function associated with the normal marginal $g_0$. Since under the normalization $\alpha_t$ all three distributions have high risk limit scenarios $(U, V)$ with $U$ and $V$ independent and $V$ standard

exponential, and since the normalizations for the vertical coordinate $V$ are determined by the vertical marginal $g_0$ which is the same in the three cases, we concentrate on the horizontal component. For the density with square level sets, $U$ is uniformly distributed on $[-1,1]$, whereas in the other two cases $U$ has a point mass at the origin. Hence, one obtains for the mixture a uniform distribution on $[-1,1]$ with a point mass of weight $2/3$ in the origin. If we use the partial compactification of the plane in [6] then it is also possible to obtain limit distributions on $[-\infty,\infty]$ for the horizontal component. There are several: A centered Gaussian density with an atom of weight $1/3$ in the origin and two atoms of weight $1/6$ in $\pm\infty$; a Laplace density with two atoms of weight $1/3$ in $\pm\infty$; an atom of weight $1/3$ or $2/3$ in the origin with the remaining mass fairly divided over the two points in $\infty$; all mass in the points in $\infty$.

Now consider an open ellipse $E$, symmetric around the diagonal, which has a unique boundary point on the line $\{x_2 = 1\}$ in $(p,1)$ for some $p \in (0,1)$. Its reflection $E'$ in the vertical axis has the boundary point $(-p,1)$. The union $S = E \cup E'$ is a star-shaped set. What does the excess measure for horizontal halfspaces look like? If we zoom in on $(p,1)$ we obtain a Gauss-exponential measure, but the measure around the point $(-p,1)$ moves off to $-\infty$. If we want weak convergence on horizontal halfspaces we have to use the normalization $\alpha_t$ above. The limit measure now lives on the two vertical lines $u = \pm p$ and has the same exponential density $e^{-v}$ on each by symmetry. $\Diamond$

If $S \subset (-1,1)^2$, and the boundary of the star-shaped set $S$ contains points on the interior of the four sides of the square, but does not contain any of the vertices, then the components $X_1$ and $X_2$ of the vector $\mathbf{X}$ with density $g$ are asymptotically independent, and so are the pairs $(-X_1, X_2)$, $(X_1, -X_2)$, $(-X_1, -X_2)$, see [3]. This also holds if $S$ is the whole square. So if $g$ is the meta density based on a heavy-tailed density in $\mathcal{D}(\rho)$, then the excess measure $\rho$ lives on the four halfaxes, and the distribution may be restricted to the region $C$.

If the high risk scenarios for horizontal halfplanes converge with the normalizations $\alpha_t(u,v) = (tu, t + a(t)v)$ of Proposition 3.14 then the limit vector $(U,V)$ has independent components and $U$ lives on the linear set $E_1$ determined by the intersection of the boundary of the limit set $E$ with the horizontal line $\{v = 1\}$. Note that in this setting the limit set $E$ is the closure of $S$, see [3].

**Proposition 3.15.** *Let $G$ be a bivariate df with marginal tails all asymptotic to the von Mises function $e^{-\psi}$ with scale function $a$. Let $\mathbf{X}^t$ denote the high risk scenario $\mathbf{X}^H$ for the horizontal halfplane $H = \{y \geq t\}$. Let $\alpha_t(u,v) = (x,y) = (tu, t + a(t)v)$. Suppose $\alpha_t^{-1}(\mathbf{X}^t) \Rightarrow \mathbf{U} = (U,V)$ for $t \to \infty$. Then $U$ and $V$ are independent, $V$ is standard exponential and $|U| \leq 1$. Let $\psi(b_n) = \log n$. If the sample clouds from $G$ scaled by $b_n$ converge onto the compact set $E$ then $\mathbb{P}\{(U,1) \in E\} = 1$.*

**Proof** The distribution of $V$ is determined by the marginal df $G_2$ and is exponential since $1 - G_2$ is asymptotic to a von Mises function, see [4], Proposition 14.1. Light tails of the marginal $G_1$ imply $(1 - G_1(e^{\epsilon}t))/(1 - G_1(t)) \to 0$ for any $\epsilon > 0$, and hence $\mathbb{P}\{U > e^{\epsilon}\} = 0$. Similarly for the left tail. Let $\sigma$ denote the excess measure extending the distribution of $(U,V)$ to $\mathbb{R}^2$; see [4], Section 14.6. Since

$\alpha_t^{-1}\alpha_{t+sa(t)} \to \beta_s$ where $\beta_s(u,v) = (u, v+s)$, it follows that $\sigma(A + (0,t)) = e^{-t}\sigma(A)$ for all Borel sets $A$ in $\mathbb{R}^2$. See Proposition 14.4 in [4]. This implies that $\sigma$ is a product measure, and hence that $U$ and $V$ are independent. Let $\beta_n(u,v) = (b_n u, b_n + b_n v)$. The sample clouds $\{\beta_n^{-1}(\mathbf{X}_1), \ldots, \beta_n^{-1}(\mathbf{X}_n)\}$ converge onto the translated set $E-(0,1)$. Now apply the additional normalization $\gamma_n(u,v) = (u, a_n v/b_n)$ where $a_n = a(b_n)$ implies $a_n/b_n \to 0$. Then $\gamma_n^{-1}\beta_n^{-1} = \alpha_n^{-1}$ where $\alpha_n(u,v) = (b_n u, b_n + a_n v)$. Hence the renormalized sample clouds converge to the Poisson point process with mean measures $\sigma$, and the restriction of $\sigma$ to $\{v \geq 0\}$ lives on $E_0 \times [0,\infty)$ where $E_0$ is the set $\{u \mid (u,1) \in E\}$. ¶

### 3.3.4 The combined situation

In a number of examples above of distributions on $C$ the high risk scenarios for horizontal halfplanes, normalized by $\alpha_t : (u,v) \mapsto (x,y) = (tu, t + a(t)v)$, converge to a random vector $(U,V)$ with independent components, $V$ is standard exponential and $U$ lives on the interval $[-1,1]$. Recall from Proposition 3.14 that for meta distributions on $D$ these high risk scenarios with the same normalization also converge to a random vector with independent components. The vertical component is again standard exponential; the horizontal component lives on the two point set $\{-1,1\}$. The associated excess measures are product measures on the vertical strip $[-1,1] \times \mathbb{R}$. The density along the vertical line is $c_0 e^{-v}/c$. Now consider the sum of these two light-tailed distributions. We can alter this sum on a compact set to make it into a probability measure. The excess measure $\tilde{\sigma}$ associated with the high risk limit distribution for this new light-tailed distribution has the same structure: It is a product measure on the vertical strip $[-1,1] \times \mathbb{R}$, the vertical component has density $ce^{-v}$, the horizontal component is a probability measure $\sigma^*$ on $[-1,1]$ with atoms $p_-$ and $p_+$ in the points $\pm 1$.

If we look at the heavy-tailed distributions we see that the excess measures $\rho$ for distributions on $D$ live on the complement of the axes, and for distributions on $C$ live on the axes. So here the sum has an excess measure which has mass both on the axes, and on the complement. The restriction of this measure to the set above the horizontal axis is characterized by the projection on the vertical coordinate with density $c\lambda/r^{\lambda+1}$ and a probability measure $\rho^*$ on the horizontal line, the spectral measure (see [4], Section 14.8) with the property that $\rho^*[c,\infty) = \rho(E)/c$, where $E = \{(u,v) \mid u \geq cv, v \geq 1\}$ is the set above the horizontal line $\{v = 1\}$ and to the right of the ray through the point $(c,1)$. The probability measure $\rho^*$ has an atom of weight $p_0$ in the origin.

**Proposition 3.16.** *The weights $p_-, p_+, p_0$ above have sum $p_- + p_0 + p_+ \geq 1$.*

**Proof** If the horizontal component $U$ on $[-1,1]$ from the high risk limit scenario due to the light-tailed density on $C$ lives on the open interval $(-1,1)$ then $p_- + p_0 + p_+ = 1$, and $p_i = c_i/c$ as in Proposition 3.14 where now $c_- = \rho((-\infty,0) \times [1,\infty))$, $c_0 = \rho(\{0\} \times [1,\infty))$, $c_+ = \rho((0,\infty) \times [1,\infty))$, and $c = c_- + c_0 + c_+ = \rho(\mathbb{R} \times [1,\infty))$. ¶

There is a simple relation between high risk scenarios for horizontal halfspaces for the heavy-tailed

density and the light-tailed meta density on $\mathbb{R}^2$ in terms of the meta transformation provided the halfspaces have the same probability mass. The inverse $K^{-1}$ maps such a $\mathbf{z}$-halfspace $H$ into an $\mathbf{x}$-halfspace $H'$. In the limit for the excess measures this yields a coordinatewise mapping from $\mathbf{z}$-space to $\mathbf{x}$-space:

$$(z_1, z_2) \mapsto (x_1, x_2) = (\mathrm{sign}(z_1), \log z_2). \tag{3.8}$$

The transformation for the horizontal coordinate is degenerate. Define the curve $\Gamma$ as the graph of $x \mapsto \mathrm{sign}(x)$ augmented with the vertical segment at the discontinuity in zero. There is a unique probability measure $\mu$ on $\Gamma$ whose horizontal projection is $\rho^*$ and whose vertical projection is $\sigma^*$. This follows from the inequality in Proposition 3.16. The excess measures $\rho$ on $(0, \infty)^2$ and $\sigma$ on $\mathbb{R}^2$ are linked by the coordinatewise exponential transformation in Proposition 3.11. The excess measures $\rho$ on $\mathbb{R} \times (0, \infty)$ and $\tilde{\sigma}$ on $[-1, 1] \times \mathbb{R}$ are linked by the coordinatewise map (3.8). The high risk limit scenarios are limits of vectors $\mathbf{X}^{H'}$ and $\mathbf{Z}^H$ where $H'$ and $H$ are corresponding horizontal halfspaces (with the same mass). These vectors are linked by the coordinatewise monotone transformation $K$ restricted to $H'$. In such a situation only a very limited class of transformations is possible in the limit, apart from affine transformations only power transformations, the exponential and its inverse the logarithmic, and four degenerate transformations amongst which the transformation sign and its inverse, see [1] Chapter 1 for details.

# 4 Discussion

In situations where chance plays a role the asymptotic description often consists of two parts, a deterministic term, catching the main effect, and a stochastic term, describing the random fluctuations around the deterministic part. Thus the average of the first $n$ observations converges to the expectation; under additional assumptions the difference between the average and the expectation, blown up by a factor $\sqrt{n}$, is asymptotically normal. Empirical dfs converge to the true df; the fluctuations are modeled by a time-changed Brownian bridge. For a positive random variable, the $n$-point sample clouds $N_n$ scaled by the $1 - 1/n$ quantile converge onto the interval $[0, 1]$ if the tail of the df is rapidly varying; if the tail is asymptotic to a von Mises function then there is a limiting Poisson point process with intensity $e^{-s}$.

Convergence to the first order deterministic term in these situations is a much more robust affair than convergence of the random fluctuations around this term. So it is surprising that for meta distributions perturbations of the original distribution which do not affect the second order fluctuations of the sample cloud at the vertices may drastically alter the shape of the limit set, the first order term. This paper tries to cast some light on the sensitivity of the meta distribution and the limit set $E$ to small perturbations of the original distribution.

Bivariate asymptotics are well expressed in terms of polar coordinates. Two points far off are close together if the angular parts are close and if the quotient of the radial parts is close to one. This geometry

is respected by certain partitions. A partition is regular if points in the same atom are uniformly close as one moves out to infinity. Call probability distributions *equivalent* if they give the same or asymptotically the same weight to the atoms of a regular partition. Equivalent distributions have the same asymptotic behaviour with respect to scaling.

This paper compares the asymptotic behaviour of a heavy-tailed bivariate density with the asymptotic behaviour of the meta density with light-tailed marginals. Small changes in the heavy-tailed density, changes which have no influence on its asymptotic behaviour, may lead to significant changes in the asymptotic behaviour of the meta distribution. We show that regular partitions for the heavy-tailed distribution and for the light-tailed meta distribution are incommensurate. The atoms at the diagonals in the light-tailed distribution fill up the quadrants for the heavy-tailed distribution; atoms at the axes in the heavy-tailed distribution fill up the four segments between the diagonals for the light-tailed distributions. Section 3.1 shows how equivalent distributions in the one world give rise to different asymptotic behaviour in the other.

In our approach the asymptotic behaviour in both worlds is investigated by rescaling. In the heavy-tailed world one obtains a limiting Poisson point process whose mean measure is an excess measure $\rho$ which is finite outside centered disks in the plane; in the light-tailed world the sample clouds converge onto a star-shaped limit set $E$. The only relation between $\rho$ and $E$ is the parameter $\lambda$. This parameter describes the rate of decrease of the heavy-tailed marginal distributions; it also is one of the two parameters which determine the shape of the limit set $E$. The measure $\rho$ describes the asymptotics for extreme order statistics; the set $E$ for the intermediate ones. In Section 3.2 it is shown that it is possible to manipulate the shape of the limit set $E$ without affecting the distribution of the extremes.

There are two worlds, the heavy-tailed and the light-tailed; the bridge linking these worlds is the meta transformation that (in our approach) maps heavy-tailed distributions into light-tailed distributions. Coordinatewise multivariate extreme value theory with its concepts of max-stable laws and exponent measures is able to cross the bridge, and to describe the asymptotic theory of the two worlds in a unified way. The exponent measures of the heavy-tailed world are linked to the light-tailed world by a coordinatewise exponential transformation.

A closer look reveals a different universe. The two worlds exist side by side like the two sides of a sheet of paper, each having its own picture. On the one side we see a landscape; on the other the portrait of a youth. Closer inspection shows an avenue leading up to a mansion in the far distance in the landscape, and a youth looking out from one of the windows; in the black pupils of his eyes on the reverse side of the paper we see a reflection of the landscape. In Section 3.3 it is shown that it is possible to disentangle the heavy- and the light-tailed parts of a distribution by using the marginals to define biregular partitions.

The heavy-tailed distribution gives information about the coordinatewise extremes. The light-tailed distribution gives this information and more: a limit shape, and limit distributions for horizontal and

vertical high risk scenarios. The limit shape is a compact star-shaped subset of the unit square; the horizontal and vertical high risk limit scenarios have independent components $(U, V)$. In appropriate coordinates, $V$ is standard exponential and $U$ is distributed over the interval $[-1, 1]$.

Biregular partitions make it possible to combine a representative of the heavy-tailed distribution (with no excess mass on the axes) with a representative of the light-tailed distribution (with asymptotically independent components) into one probability distribution. The excess measure of the combined distribution is the sum of the two individual excess measures; the limit set is the union of the two individual limit sets. The biregular partition is finer than the regular partition for the light-tailed distribution. It not only sees the limit set, it is fine enough to discern the high risk asymptotics in all directions.

While the asymptotics of heavy-tailed multivariate distributions is well understood and nicely reflected in the asymptotic behaviour of the copula at the vertices of the unit cube, for light-tailed distributions there are still many open areas. The possible high risk limit scenarios for light-tailed distributions have been described; see [4], Section 14. However, their domains of attraction are still unexplored territory. This paper shows that the asymptotics of heavy-tailed multivariate distributions do not suffice to describe the asymptotic behaviour of the light-tailed meta distributions.

One possible explanation for the loss of information in crossing the bridge between heavy and light tails is the highly non-linear nature of the meta transformation. Coordinate hyperplanes are preserved, and so are centered coordinate cubes because of the equal and symmetric marginals. Geometric concepts such as sphere, convex set, halfspace and direction are lost. The meta transformation maps rays in the $\mathbf{x}$-world which do not lie in one of the diagonal hyperplanes $\{x_i = x_j\}$ into curves which are asymptotic to the halfaxis in the center of the sector bounded by the diagonal planes. Similarly, the inverse transformation maps rays which do not lie in one of the coordinate planes into curves converging to the diagonal ray in the center of the orthant bounded by the coordinate planes. In the bivariate setting there is a clear duality between diagonals and axes. Heavy-tailed asymptotic dependence reduces to co- or counter-monotonicity in the light-tailed world; horizontal and vertical high risk limit scenarios in the light-tailed world reduce to asymptotic independence in the heavy-tailed world. Biregular partitions allow us to see the combined effect, by describing the two worlds side by side.

The asymptotics of light-tailed densities with given marginals is not well understood. We hope to return to this topic later. Similarly it is not clear what role is played by the remainder set $O$ in our decomposition $C \cup D \cup O$.

# A Appendix

## A.1 Supplementary results

**Lemma A.1.** *Let $g$ be a positive continuous symmetric density which is asymptotic to a von Mises function $e^{-\psi}$. There exists a continuous unimodal symmetric density $g_1$ such that for all $c \in (0,1)$*

$$g_1(s)/g(s) \to 0 \qquad g_1(cs)/g(s) \to \infty \qquad s \to \infty.$$

**Proof** Let $M_n(s) = \psi(s) - \psi(s - s/n)$, and let $M_n^*(s) = \min_{t>s} M_n(t)$ for $n \geq 2$. Each function $M_n^*$ is increasing, continuous and unbounded (since $t/a(t) \to \infty$), and for each $s > 0$ the sequence $M_n^*(s)$ is decreasing. There exists a continuous increasing unbounded function $b$ such that

$$\lim_{s\to\infty} M_n(s) - b(s) = \infty \qquad n = 1, 2, \ldots. \tag{A.1}$$

Indeed, define $M^*(s) = M_n^*(s)$ on $[a_n, b_n] = \{M_n^* \in [n, n+1]\}$, and $M^*(s) = n$ on $[b_{n-1}, a_n]$. Then $M^*(s)$ is increasing and $M^*(s) \leq M_n^*(s)$ eventually for each $n \geq 2$. Set $b(s) = M^*(s)/2$ to obtain (A.1). The function $g_1 = e^{-\psi_1}$ with $\psi_1(s) = \psi(s) + b(s)$ is decreasing on $[0, \infty)$ and continuous, and $b(s) \to \infty$ implies $g_1(s)/g(s) \to 0$ for $s \to \infty$. For $c = 1 - 1/m$ the relations

$$\psi(s) - \psi_1(cs) = \psi(s) - \psi(cs) - b(cs) \geq M_m(s) - b(s) \to \infty$$

hold, and yield the desired result. ¶

**Proposition A.2.** *Let $g_d$ be a continuous positive symmetric density which is asymptotic to a von Mises function $e^{-\psi}$. Choose $r_n$ such that $\int_{r_n}^{\infty} g_d(s)ds \sim 1/n$. Let $g_1$ be the probability density in Lemma A.1. There exists a unimodal density $g(\mathbf{x}) = g_*(\|\mathbf{x}\|_\infty)$ on $\mathbb{R}^d$ with cubic level sets and marginals $g_1$. The sample clouds from the density $g$, scaled by $r_n$ converge onto the standard cube $[-1,1]^d$. The functions $h_n(\mathbf{u}) = nr_n^d g(r_n\mathbf{u})$ are unimodal with cubic level sets. They satisfy*

$$h_n(\mathbf{u}) \to \begin{cases} \infty & \mathbf{u} \in (-1,1)^d \\ 0 & \mathbf{u} \notin [-1,1]^d. \end{cases}$$

*Let $E$ be a closed subset of $C = [-1,1]^d$, containing the origin as interior point, star-shaped with continuous boundary. Set $c_E = |E|/2^d$. Then $g_E(\mathbf{x}) = g_*(n_E(\mathbf{x}))/c_E$ is a probability density, and the sample clouds from $g_E$ scaled by $r_n$ converge onto the set $E$.*

**Proof** Existence of $g$ follows from Proposition A.3 in [2]. Let $G_1$ be the df with density $g_1$. Then $n(1 - G_1(cr_n)) \to 0$ for $c > 1$, and $n(1 - G_1(cr_n)) \to \infty$ for $c \in (0,1)$. Let $\pi$ be the probability distribution with the unimodal density $g$. The limit relations on the marginal dfs $G_1$ imply that $n\pi(B_n) \to \infty$ for the block $B_n = [-2r_n, 2r_n]^{d-1} \times [cr_n, 2r_n]$ for any $c \in (0,1)$. Since $h_n$ is unimodal with cubic level sets it follows that $h_n \to \infty$ uniformly on $[-c, c]^d$ for any $c \in (0,1)$. (Since $h_n(c\mathbf{1}) \leq k$ implies $n\pi(B_n) \leq (2c)^d k$.)

The area of a horizontal slice of the density $g_E$ at level $y/c_E > 0$ is less than the area of the horizontal slice of $g$ at level $y$, but the height of the slice is proportionally more by the factor $c_E$. So the slices have the same volume. The level sets of the scaled densities are related:

$$\{h_E \geq t/c_E\} = rE \qquad \Longleftrightarrow \qquad \{h \geq t\} = rC.$$

So the function $h_E$ mimics the behaviour of $h = h_C$. ¶

## A.2 Summary of notation

This section is intended to provide a guide to the notation. Throughout the paper, it is convenient to keep in mind two spaces: $\mathbf{z}$-space on which the heavy-tailed dfs $F, F^*, \ldots$ are defined, and $\mathbf{x}$-space on which the light-tailed dfs $G, G^*, \ldots$ are defined. Table A.1 compares notation used for mathematical objects on these two spaces. Table A.2 can be consulted while reading Section 3 in order to keep track of various symbols used to distinguish original and meta distributions with certain properties and purpose.

Table A.1: Symbols used to distinguish various objects of interest in $\mathbf{z}$-space and in $\mathbf{x}$-space. Notation for marginals assumes that all marginal densities are equal and symmetric as in (3.1).

| $\mathbf{z}$-space | $\mathbf{x}$-space | Comments |
|---|---|---|
| $F$ and $f$ | $G = F \circ K$ and $g$ | joint df and density |
| $F_0$ and $f_0$ | $G_0 = F_0 \circ K_0$ and $g_0$ | marginal df and density |
| $\pi$ | $\mu$ | probability measures |
| $\pi_n$ | $\mu_n$ | mean measures of scaled sample clouds |
| $\mathbf{Z}$, $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ | $\mathbf{X}$, $\mathbf{X}_1, \mathbf{X}_2, \ldots$ | random vectors |
| $N_n$ | $M_n$ | scaled $n$-point sample clouds |
| $N$: Poisson point process | $E$: limit set | limit of scaled sample clouds |
| $c_n :\ 1 - F_0(c_n) \sim 1/n$ | $b_n :\ -\log(1 - G_0(b_n)) \sim \log n$ | scaling constants |
| $(B_n)$, $t_n = K_0(s_n)$ | $(A_n)$, $s_n$ | block partitions, division points |

Table A.2: Symbols used in Section 3 to discuss robustness and sensitivity properties of meta distributions. Where not explicitly mentioned, notation for the associated meta df is analogous, e.g. $\tilde{G} = \tilde{F} \circ K$ and similarly for other objects $\tilde{f}$, $\tilde{E}$, etc.

| Symbols | Description |
|---|---|
| $F$, $G$ | dfs satisfying the assumptions of the standard set-up |
| $F^*$, $G^*$ | dfs constructed to have properties (P.1)-(P.4) and yield the same asymptotics for $F$ and $F^*$, or $G$ and $G^*$; |
| $\hat{F}$ | any original df with marginals $F_0$ |
| $\tilde{F}$ | df obtained by changing $\hat{F}$ to have marginals $\tilde{F}_j$ tail equivalent to $F_0$ (e.g. mixtures in Section 3.2) |
| $F^o$ | original df with lighter marginals than $F$; see (3.2) |

Table A.3: Miscellaneous symbols.

| Symbol | Description |
|---|---|
| $f \asymp \tilde{f}$ | ratios $f(\mathbf{x})/\tilde{f}(\mathbf{x})$ and $\tilde{f}(\mathbf{x})/f(\mathbf{x})$ are bounded eventually for $\|\mathbf{x}\| \to \infty$ |
| $f \sim \tilde{f}$ | $\tilde{f}(\mathbf{x})/f(\mathbf{x}) \to 1$ for $\|\mathbf{x}\| \to \infty$ |
| $n_D$ | *gauge function* of set $D$: $n_D(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, $D = \{n_D < 1\}$ and $n_D(c\mathbf{x}) = cn_D(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^d$, $c \geq 0$ |
| $\mathbf{e}$ | a vector of ones in $\mathbb{R}^d$ |
| $B$ | the open Euclidean unit ball in $\mathbb{R}^d$ |

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